Please check the examination details below before entering your candidate information

Candidate surname	Other names

Pearson Edexcel Level 3 GCE

Centre Number	Candidate Number

Thursday 08 October 2020

Afternoon (Time: 1 hour 30 minutes) | Paper Reference **9FM0/02**

Further Mathematics

Advanced

Paper 2: Core Pure Mathematics 2

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 7 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over

1. The curve C has equation

$$y = 31\sinh x - 2\sinh 2x \qquad x \in \mathbb{R}$$

Determine, in terms of natural logarithms, the exact x coordinates of the stationary points of C.

(7)

remembering from Pure Yr 1 that to find the stationary points of a curve means finding $\frac{dy}{dx}=0$ - but here we are dealing with a Hyperbolic curve C

first differentiating the curve - using $\frac{d}{dx}(\sinh kx) = k\cosh kx$

$$\frac{dy}{dx} = 31\cosh x - 4\cosh 2x$$

and making it equal 0

$31\cosh x - 4\cosh 2x = 0$, giving us a hyperbolic equation which we need to solve

METHOD 1: using cosh double angle identity

first dealing with the cosh double angle - know that:

$$\cos 2x = 1 - 2\sin^2 x \text{ or } 2\cos^2 x - 1$$

$$\Rightarrow \cosh 2x = 1 + 2\sinh^2 x \text{ or } 2\cosh^2 x - 1$$

Osborne's rule

...choosing the latter to match the linear '$31\cosh x$' term in the equation; equation now becomes:

$$31\cosh x - 4(2\cosh^2 x - 1) = 0$$

expand brackets

$$31\cosh x - 8\cosh^2 x + 4 = 0$$

$$\Rightarrow 8\cosh^2 x - 31\cosh x - 4 = 0$$

notice this is a quadratic in cosh x - using substitution: $y = \cosh x$

$8y^2 - 31y - 4 = 0$ - evaluate on calc equation solver or 'ac' method

$ac = -32$, $b = -31$ } need two numbers that multiply to give -32 and sum to give -31 : $1, -32$

$$8y^2 + y - 32y - 4 = 0$$

$$y(8y+1) - 4(8y+1) = 0$$

$\Rightarrow (y-4)(8y+1)=0$

Subbing the substitution back in: $(\cosh x-4)(8\cosh x+1)=0$

making each bracket equal 0:

$8\cosh x = -1$

$\div 8 \qquad \div 8$

$\cosh x = -1/8$

but know from the properties of the $y=\cosh x$ graph that $\cosh x > 0$ $\therefore$ reject

left with $\cosh x = 4$

↳ two main ways to solve this

WAY 1: using definition of inverse cosh

taking arcosh of both sides

$x = \text{arcosh}(4)$

using formula booklet definition for inverse cosh: $\text{arcosh}\,x = \ln(x+\sqrt{x^2-1})$

where $x \to 4$

$x = \ln(4 \pm \sqrt{4^2-1})$

$\Rightarrow \boxed{x = \ln(4 \pm \sqrt{15})}$

WAY 2: using exponential definition coshx

$\cosh x = \frac{1}{2}(e^x+e^{-x})$

$\frac{1}{2}(e^x+e^{-x}) = 4$

$\times 2 \qquad \times 2$

$e^x + e^{-x} = 8$

$\times e^x \qquad \times e^x$

$e^{2x}+1 = 8e^x$

$e^{2x} - 8e^x + 1 = 0$

noticing this is a quadratic in e^x:

using substitution $y = e^x$

$y^2 - 8y + 1 = 0$

↳ using calc equation solver or quadratic formula

$y = \frac{8 \pm \sqrt{(-8)^2-4(1)(1)}}{2}$

$= \frac{8 \pm \sqrt{64-4}}{2}$

$= \frac{8 \pm \sqrt{60}}{2} = \frac{8 \pm \sqrt{15 \times 4}}{2}$

$= \frac{8 \pm 2\sqrt{15}}{2}$

$\Rightarrow y = 4 \pm \sqrt{15}$

subbing in $y = e^x$

$e^x = 4 \pm \sqrt{15}$

taking ln of both sides

$\boxed{x = \ln(4 \pm \sqrt{15})}$

Question 1 continued

METHOD 2: using coshx hyperbolic definition

$$31\left(\tfrac{1}{2}(e^{x}+e^{-x})\right) - 4\left(\tfrac{1}{2}(e^{2x}+e^{-2x})\right) = 0$$

$$= \frac{31}{2}(e^{x}+e^{-x}) - 2(e^{2x}+e^{-2x}) = 0$$

$\times 2$ $\times 2$

$$\Rightarrow 31(e^{x}+e^{-x}) - 4(e^{2x}+e^{-2x}) = 0$$

expand brackets and evaluate index powers

$$31e^{x} + \frac{31}{e^{x}} - 4e^{2x} - \frac{4}{e^{2x}} = 0$$

$\times e^{2x}$ $\times e^{2x}$

$$\Rightarrow 31e^{3x} + 31e^{x} - 4e^{4x} - 4 = 0$$

taking all terms to RHS

$$4e^{4x} - 31e^{3x} - 31e^{x} + 4 = 0$$

notice this is a quartic in e^{x} - use the substitution $y = e^{x}$

$$4y^{4} - 31y^{3} - 31y + 4 = 0$$

calc equation solver

only real roots:

$$y = 4 \pm \sqrt{15}$$

subbing e^{x} back in

$$e^{x} = 4 \pm \sqrt{15}$$

taking logs of both sides

$$\boxed{x = \ln(4 \pm \sqrt{15})}$$

(Total for Question 1 is 7 marks)

2. In an Argand diagram, the points A and B are represented by the complex numbers $-3+2\text{i}$ and $5-4\text{i}$ respectively. The points A and B are the end points of a diameter of a circle C.

(a) Find the equation of C, giving your answer in the form

$$|z-a| = b \qquad a \in \mathbb{C},\ b \in \mathbb{R}$$

(3)

The circle D, with equation $|z-2-3\text{i}| = 2$, intersects C at the points representing the complex numbers z_1 and z_2

(b) Find the complex numbers z_1 and z_2

(6)

(a) know from Core Pure Yr 1 that if a loci of points is given in the form

$|z-a| = b$

then it represents a circle, centre 'a' and radius 'b'

· therefore first finding the centre of this circle by finding the midpoint of the two end points of the diameter of C i.e A and B

↳ same as finding the average of 'x' and 'y' coordinates

$M\left(\frac{-3+5}{2}, \frac{2+-4}{2}\right) \therefore M(1,-1)$

and using fact that coordinates (a,b) are used to represent the complex number a+bi

$\Rightarrow a = 1-i$

· next finding radius - know centre of circle, so finding distance from centre to any of A or B - using $d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

...with A:

$d = \sqrt{(-3-1)^2 + (2--1)^2}$

$= \sqrt{16+9} = \sqrt{25} = 5$

...with B:

$d = \sqrt{(5-1)^2 + (-4--1)^2}$

$= \sqrt{(4)^2 + (-3)^2}$

$= \sqrt{16+9} = \sqrt{25} = 5$

$\Rightarrow b = 5$

subbing these into general loci equation

$\Rightarrow |z-(1-i)| = 5$

$\Rightarrow \boxed{z-1+i = 5}$

Question 2 continued

(b) question is basically asking us to find p.o.i between two circles – the best way to do this is to solve the Cartesian equations simultaneously

↳ know centre of C as $(1,-1)$ and rad. $=5$

∴ using Cartesian equation of a circle : $(x-x_1)^2+(y-y_1)^2=r^2$

$$C:(x-1)^2+(y-1)^2=25 \quad -①$$

now for D – rewrite its general loci equation form such that can read its centre and radius straight from it :

$$|z-(2+3i)|=2 \Rightarrow \text{centre}=(2,3), \ \text{radius}=2$$

subbing this into the Cartesian equation of a circle

$$D:(x-2)^2+(y-3)^2=2^2$$

$$\Rightarrow D:(x-2)^2+(y-3)^2=4 \quad -②$$

need to solve ① and ② simultaneously ∴ expanding both equations

...for C :

$$x^2-2x+1+y^2+2y+1=25$$

$$\Rightarrow x^2+y^2-2x+2y=23 \quad -①$$

...for D :

$$x^2-4x+4+y^2-6y+9=4$$

$$x^2+y^2-4x-6y=-9 \quad -②$$

solving ① and ② simultaneously

①-②

$$\begin{array}{r} x^2+y^2-2x+2y=23 \\ -\quad x^2+y^2-4x-6y=-9 \\ \hline 2x+8y=32 \end{array} \quad -③$$

from ③ can either make 'x' the subject and sub into ① or ② or make 'y' the subject and sub into ① or ②

WAY 1: make 'x' the subject

$$2x=32-8y$$

÷2 ÷2

$$x=16-4y$$

sub into any of ① or ②, eg. ①

$$(16-4y)^2+y^2-2(16-4y)+2y=23$$

expand brackets

$$256-128y+16y^2+y^2-32+8y+2y=23$$

collect like terms

WAY 2: make 'y' the subject

$$8y=32-2x$$

÷8 ÷8

$$y=\frac{\cancel{32}^{16}-\cancel{2}x}{\cancel{8}_4}=\frac{16-x}{4}$$

sub into any of ① or ② eg. ①

$$x^2+\left(\frac{16-x}{4}\right)^2-2x+2\left(\frac{16-x}{4}\right)=23$$

expand brackets

Question 2 continued

$17y^2 - 118y + 201 = 0$

calc equation solver

$y = \frac{67}{17}, 3$

sub into ③ to get 'x'

- when $y = \frac{67}{17}$,

$x = 16 - 4\left(\frac{67}{17}\right)$

$= \frac{4}{17} \Rightarrow \left(\frac{4}{17}, \frac{67}{17}\right)$

- when $y = 3$,

$x = 16 - 4(3)$

$= 16 - 12 = 4$

$\Rightarrow (4, 3)$

using fact that (a, b) represents complex number $a + bi$

$\Rightarrow \boxed{z_1 = \frac{4}{17} + \frac{67}{17}i \quad z_2 = 4 + 3i}$

$x^2 + \frac{(16-x)^2}{16} - 2x + \frac{16-x}{2} = 23$

×16 ×16

$16x^2 + (16-x)^2 - 32x + 8(16-x) = 368$

expand brackets

$16x^2 + 256 - 32x + x^2 - 32x + 128 - 8x = 368$

collect like terms

$17x^2 - 72x + 16 = 0$

calc equation solver

$\Rightarrow x = \frac{4}{17}$ or 4

- when $x = \frac{4}{17}$, sub into ③ for 'y':

$y = \frac{16 - \frac{4}{17}}{4} = \frac{67}{17}$

$\Rightarrow \left(\frac{4}{17}, \frac{67}{17}\right)$

- when $x = 4$,

$y = \frac{16 - 4}{4} = \frac{12}{4} = 3$

$\Rightarrow (4, 3)$

using fact that (a, b) represent complex numbers '$a + bi$'

$\Rightarrow \boxed{z_1 = \frac{4}{17} + \frac{67}{17}i \quad z_2 = 4 + 3i}$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

Question 2 continued

(Total for Question 2 is 9 marks)

3. A scientist is investigating the concentration of antibodies in the bloodstream of a patient following a vaccination.
The concentration of antibodies, x, measured in micrograms (µg) per millilitre (ml) of blood, is modelled by the differential equation

$$100\frac{d^2x}{dt^2} + 60\frac{dx}{dt} + 13x = 26$$

where t is the number of weeks since the vaccination was given.

(a) Find a general solution of the differential equation.

(4)

Initially,

- there are no antibodies in the bloodstream of the patient
- the concentration of antibodies is estimated to be increasing at 10 µg/ml per week

(b) Find, according to the model, the maximum concentration of antibodies in the bloodstream of the patient after the vaccination.

(8)

A second dose of the vaccine has to be given to try to ensure that it is fully effective.
It is only safe to give the second dose if the concentration of antibodies in the bloodstream of the patient is less than 5 µg/ml.

(c) Determine whether, according to the model, it is safe to give the second dose of the vaccine to the patient exactly 10 weeks after the first dose.

(2)

(a) noticing this is a non-homogenous 2ODE

$\boxed{A.E}$ $100m^2 + 60m + 13 = 0$

calc equation solver or quadratic formula

$$m = \frac{-60 \pm \sqrt{(60)^2 - 4(100)(13)}}{2(100)}$$

$$= \frac{-60 \pm \sqrt{-1600}}{200} = \frac{\overset{-3}{\cancel{-60}} \pm \overset{2}{\cancel{40}}i}{\underset{10}{\cancel{200}}} = \frac{-3 \pm 2i}{10} = -0.3 \pm 0.2i$$

notice the solutions to the A.E are in the form $\alpha \pm \beta i$ ∴ substituting into corresponding C.F formula : $x = e^{\alpha t}(A\cos\beta t + B\sin\beta t)$

$\Rightarrow$ C.F: $x = e^{-0.3t}(A\cos 0.2t + B\sin 0.2t)$

now for P.I looking at table

Form of f(x)	Form of particular integral
k	λ
$ax + b$	$\lambda + \mu x$
$ax^2 + bx + c$	$\lambda + \mu x + \nu x^2$
ke^{px}	λe^{px}
$m\cos\omega x$	$\lambda\cos\omega x + \mu\sin\omega x$
$m\sin\omega x$	$\lambda\cos\omega x + \mu\sin\omega x$
$m\cos\omega x + n\sin\omega x$	$\lambda\cos\omega x + \mu\sin\omega x$

WARNING!

let $x = \lambda$

$\frac{dx}{dt} = 0$

Question 3 continued $\frac{d^2x}{dt^2}=0$

subbing into 2ODE

$100(0)+60(0)+13\lambda=26$

$\Rightarrow 13\lambda=26$

$\div 13 \quad \div 13$

$\boxed{\lambda=2}$

and following rule that G.S = C.F + P.I

G.S : $\boxed{x=e^{-0.3t}(A\cos 0.2t+B\sin 0.2t)+2}$

(b) considering 'initial conditions':

at $t=0$, $x=0$ sub into above G.S

$0=e^{-0.3(0)}(A\cos(0.2\times 0)+B\sin(0.2\times 0))+2$

$\Rightarrow 0=A+2 \Rightarrow A=-2$

at $t=0$, $\frac{dx}{dt}=10$

differentiating G.S using product rule and $\frac{d}{dt}e^{kt}=ke^{kt}$,

$\frac{d}{dt}(\sin kt)=k\cos kt$ and $\frac{d}{dt}(\cos kt)=-k\sin kt$

$\frac{dx}{dt}=-0.3e^{-0.3t}(-2\cos 0.2t+B\sin 0.2t)+e^{-0.3t}(-0.2(-2)\sin 0.2t+0.2B\cos 0.2t)$

sub in initial condition

$10=-0.3e^{-0.3(0)}(-2\cos(0.2\times 0)+B\sin(0.2\times 0))+e^{-0.3(0)}(0.4\sin(0.2\times 0)$
$+0.2B\cos(0.2\times 0)$

$10=-0.3(-2)+0.2B$

$10=0.6+0.2B$

$\Rightarrow 0.2B=9.4$

$\div 0.2 \quad \div 0.2$

$B=47$

subbing into G.S:

$\boxed{P.I}$ $x=e^{-0.3t}(-2\cos 0.2t+47\sin 0.2t)+2$

now using from Pure Yr 1 that max. concentration occurs where $\frac{dx}{dt}=0$

Question 3 continued

now differentiate P.I using product rule and equate to 0

$$\frac{dx}{dt} = -0.3e^{-0.3t}(-2\cos 0.2t + 47\sin 0.2t) + e^{-0.3t}(0.4\sin 0.2t + 9.4\cos 0.2t) = 0$$

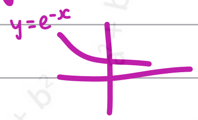
solving above for the 't' at which the max. concentration occurs - factorise $e^{-0.3t}$ out

$$e^{-0.3t}\left[\cos(0.2t)(0.6+9.4) + \sin(0.2t)(-14.1+0.4)\right] = 0$$

$$\Rightarrow e^{-0.3t}\left[10\cos 0.2t - 13.7\sin 0.2t\right] = 0$$

making each bracket equal 0

$e^{-3t} \neq 0$ due to exponential graph properties



$\therefore$ reject

$\therefore$ left with $10\cos 0.2t - 13.7\sin 0.2t = 0$

$\Rightarrow 10\cos 0.2t = 13.7\sin 0.2t$

$\div \cos 0.2t$

$10 = \dfrac{13.7\sin 0.2t}{\cos 0.2t}$

$\Rightarrow 13.7\tan 0.2t = 10$

$\div 13.7 \qquad \div 13.7$

$\Rightarrow \tan 0.2t = \dfrac{10}{13.7}$

$\Rightarrow 0.2t = \tan^{-1}\left(\dfrac{10}{13.7}\right)$

$= 0.630.$

$\div 0.2$

$\Rightarrow t = 3.152\ldots$ weeks

subbing this 't' into G.S

$$x = e^{-0.3(3.152\ldots)}\left(47\sin(0.2\times 3.152\ldots) - 2\cos(0.2\times 3.152\ldots)\right) + 2$$

$$\boxed{\Rightarrow x_{max} = 12.1\ \mu g/ml}$$

(c) need to see if the concentration of antibodies at $t=10$ is $<$ or $>$ 5

$$x = e^{-0.3(10)}\left(-2\cos(0.2\times 10) + 47\sin(0.2\times 10)\right) + 2$$

$$= e^{-3}(-2\cos(2) + 47\sin 2) + 2$$

$$= 4.16.. < 5$$

Question 3 continued

=) it would be safe to give second dose //

(Total for Question 3 is 14 marks)

4. (a) Use de Moivre's theorem to prove that

$$\sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$$

(5)

(b) Hence find the distinct roots of the equation

$$1 + 7x - 56x^3 + 112x^5 - 64x^7 = 0$$

(5)

giving your answer to 3 decimal places where appropriate.

(a) question is asking us to convert multi-angle trig expression to high trig powers

$\therefore$ following usual steps:

step 1: rewrite LHS of the equation as a power to then evaluate it using DMT on the RHS

$$(\cos\theta + i\sin\theta)^7 = \cos 7\theta + i\sin 7\theta$$

step 2: evaluate the Binomial expansion on the LHS

$$(\cos\theta)^7 + \binom{7}{1}(\cos\theta)^6(i\sin\theta)^1 + \binom{7}{2}(\cos\theta)^5(i\sin\theta)^2 + \binom{7}{3}(\cos\theta)^4(i\sin\theta)^3$$
$$+ \binom{7}{4}(\cos\theta)^3(i\sin\theta)^4 + \binom{7}{5}(\cos\theta)^2(i\sin\theta)^5 + \binom{7}{6}(\cos\theta)(i\sin\theta)^6 + (i\sin\theta)^7$$

evaluate choose function on calc and simplify

$i^1 = i$
$i^2 = -1$
$i^3 = -i$
$i^4 = 1$

$$= \cos^7\theta + 7i\cos^6\theta\sin\theta - 21\cos^5\theta\sin^2\theta - 35i\cos^4\theta\sin^3\theta + 35\cos^3\theta\sin^4\theta$$
$$+ 21i\cos^2\theta\sin^5\theta - 7\cos\theta\sin^6\theta - i\sin^7\theta$$

step 3:

but the question only asks for the $\sin 7\theta$ $\therefore$ compare imaginary parts

$$\sin 7\theta = 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta$$

and need RHS in terms of powers of sin $\therefore$ replace all cos with main Pythagorean identity rearranged: $\cos^2\theta = 1 - \sin^2\theta$

$$= 7(1-\sin^2\theta)^3\sin\theta - 35(1-\sin^2\theta)^2\sin^3\theta + 21(1-\sin^2\theta)\sin^5\theta$$
$$- \sin^7\theta$$

have to expand brackets

$$= 7(1 - 3\sin^2\theta + 3\sin^4\theta - \sin^6\theta)\sin\theta - 35(1 - 2\sin^2\theta + \sin^4\theta)\sin^3\theta$$
$$+ 21(1-\sin^2\theta)\sin^5\theta - \sin^7\theta$$

Question 4 continued

$= 7\sin\theta - 21\sin^3\theta + 21\sin^5\theta - 7\sin^7\theta - 35\sin^3\theta + 70\sin^5\theta - 35\sin^7\theta$
$+ 21\sin^5\theta - 21\sin^7\theta - \sin^7\theta$

collect like powers

$\boxed{\therefore \sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta}$

(b) notice how the equation in (b) is formed by making the substitution: $x = \sin\theta$ into equation for $\sin 7\theta$ and adding 1 onto the LHS

$\sin 7\theta + 1 = 7x - 56x^3 + 112x^5 - 64x^7 = 0$

need to solve $\sin 7\theta + 1 = 0$

$\Rightarrow \sin 7\theta = -1$

$7\theta = \sin^{-1}(-1)$

WAY 1: evaluate in degrees (in green indicated use of sin angle law)

$7\theta = -90°, \ 180° - (-90°) = 270°$

$-360°$

$-450°, \ 180° - (-450°) = 630°$

$\Rightarrow 7\theta = -450°, -90°, 270°, 630°$

$\div 7$ $\div 7$

$\theta = \frac{-450°}{7}, \frac{-90°}{7}, \frac{270°}{7}, \frac{630°}{7}$

subbing $x = \sin\theta$ back in

$x = \sin\theta = \sin\left(\frac{-450°}{7}\right), \sin\left(\frac{-90°}{7}\right), \sin\left(\frac{270°}{7}\right), \sin\left(\frac{630°}{7}\right)$

$= -0.90096\ldots, -0.22252\ldots, 0.6234\ldots$

$\boxed{\therefore -0.901, -0.223, 0.623, 1 \ (3\text{d.p})}$

WAY 2: radians

$7\theta = -\pi/2, \ (\pi - -\pi/2) = 3\pi/2$

(-2π)

$-\frac{5\pi}{2}, \ \left(\pi - \left(-\frac{5\pi}{2}\right)\right) = \frac{7\pi}{2}$

$\Rightarrow 7\theta = -\frac{5\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}$

$\div 7$ $\div 7$

$\theta = -\frac{5\pi}{14}, -\frac{\pi}{14}, \frac{3\pi}{14}, \frac{7\pi}{14}$

Question 4 continued

Subbing in $x = \sin\theta$

$x = \sin\theta = \sin\left(-\frac{5\pi}{14}\right), \sin\left(-\frac{\pi}{14}\right), \sin\left(\frac{3\pi}{14}\right), \sin\left(\frac{7\pi}{14}\right)$

$\boxed{= -0.901, -0.223, 0.623, 1 \text{ (3 d.p)}}$

Question 4 continued

(Total for Question 4 is 10 marks)

5. (a)

$$y = \tan^{-1} x$$

Assuming the derivative of $\tan x$, prove that

$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

(3)

$$f(x) = x\tan^{-1} 4x$$

(b) Show that

$$\int f(x)\,dx = Ax^2\tan^{-1} 4x + Bx + C\tan^{-1} 4x + k$$

where k is an arbitrary constant and A, B and C are constants to be determined.

(5)

(c) Hence find, in exact form, the mean value of $f(x)$ over the interval $\left[0, \frac{\sqrt{3}}{4}\right]$

(2)

(a)

$$y = \tan^{-1}(x)$$

taking tan of both sides

$$\tan y = \tan\tan^{-1}(x)$$

$$\Rightarrow \tan y = x$$

differentiate implicitly - assuming $\frac{d}{dx}(\tan x) = \sec^2 x$

$$\sec^2 y \frac{dy}{dx} = 1$$

and rearrange for $\frac{dy}{dx}$

$\div \sec^2 y$ $\div \sec^2 y$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

but need expression in terms of 'x' - use the identity : $\sec^2 y = 1 + \tan^2 y$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+\tan^2 y} = \boxed{\frac{1}{1+x^2}}$$

(b) see straight away that we're asked to integrate the product of a linear 'x' term and an inverse trig function - hints at need for I.B.P

...using pneumonic for what should be the 'u':

Question 5 continued

Logs
Inverse trig functions (✓)
Algebraic expressions
Trig
Exponentials

$u = \tan^{-1}(4x)$

differentiate - using chain rule on result in part (a)

$u' = \frac{1}{1+(4x)^2} \times 4$ $\qquad v = x$

$= \frac{4}{1+16x^2}$ $\qquad v' = \frac{x^2}{2}$

and using IBP : $\int uv\,dx = uv - \int u'v\,dx$

$$\Rightarrow \int x\tan^{-1}(4x)\,dx = \frac{x^2}{2}\tan^{-1}(4x) - \int \frac{4}{1+16x^2}\left(\frac{x^2}{2}\right)dx$$

factorise ½ out

$$\int x\tan^{-1}(4x)\,dx = \frac{x^2}{2}\tan^{-1}(4x) \boxed{-\frac{1}{2}\int \frac{4x^2}{1+16x^2}\,dx}$$

consider $-\frac{1}{2}\int \frac{4x^2}{1+16x^2}\,dx$

↳ from the ways to integrate a fractional expression: (explained more in detail on pg.21 - end of question)

Fractional expressions

4a. Can I split the numerator?
Is there a single term in the denominator?
4b. Can I do partial fractions?
Does the denominator factorise?
4c. Can I do algebraic division?
Is the fraction improper?

METHOD 1 : could manipulate the inside of the integral to then 'split the numerator' - 2 separate fractions

$$= -\frac{1}{8}\int \frac{16x^2+1-1}{16x^2+1}\,dx = -\frac{1}{8}\int\left(1 - \frac{1}{16x^2+1}\right)dx$$

split integral

$$= -\frac{1}{8}\int 1\,dx + \boxed{\frac{1}{8}\int \frac{1}{16x^2+1}}\,dx$$

looking at formula booklet to see which standard result to use

$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\ln \cosh x$
$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\left(\frac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{x^2-a^2}}$	$\operatorname{arcosh}\left(\frac{x}{a}\right),\ \ln\{x+\sqrt{x^2-a^2}\} \quad (x > a)$
$\frac{1}{\sqrt{a^2+x^2}}$	$\operatorname{arsinh}\left(\frac{x}{a}\right),\ \ln\{x+\sqrt{x^2+a^2}\}$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert = \frac{1}{a}\operatorname{artanh}\left(\frac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert$

$$= \frac{1}{8} \times \frac{1}{16} \int \frac{1}{x^2 + 1/16} \, dx$$

$$a^2 = 1/16$$
$$a = 1/4$$

$$= \frac{1}{128} \left[\frac{1}{1/4} \arctan\left(\frac{x}{1/4}\right) \right] + k$$

$$= \frac{1}{128} \left[4 \arctan(4x) \right] + k$$

$$= \frac{1}{32} \arctan(4x) + k$$

and finally subbing all into I.B.P

$$\int x \tan^{-1}(4x) \, dx = \boxed{\frac{x^2}{2} \tan^{-1}(4x) - \frac{1}{8} x + \frac{1}{32} \arctan(4x) + k}$$

$\Rightarrow A = 1/2, \; B = -1/8, \; C = 1/32$

WAY 2: integration by substitution

consider $\int \frac{4x^2}{1+16x^2} \, dx$

let $\tan u = 4x$

$\Rightarrow u = \tan^{-1}(4x)$

$\Rightarrow x = \frac{\tan u}{4}$

$\frac{du}{dx}$ = using chain rule on inverse trig result for (b)

$$= \frac{4}{1+16x^2}$$

$$\Rightarrow \frac{dx}{du} = \frac{1+16x^2}{4}$$

$$\Rightarrow dx = \frac{1+16x^2}{4} du$$

sub into integral

$$\int \frac{4x^2}{1+\cancel{16x^2}} \left(\frac{1 \cancel{+16} x^2}{\cancel{4}} \right) du$$

$$= \int x^2 \, du$$

$$= \int \left(\frac{\tan u}{4} \right)^2 du$$

$$= \frac{1}{16} \int \tan^2 u \, du$$

from Pure Yr 1 - don't know standard result for $\int \tan u \, du$ $\therefore$ using identity: $\tan^2 u = \sec^2 u - 1$

$$= \frac{1}{16} \int \sec^2 u - 1 \, du$$

integrate using $\int \sec^2 u\,du = \tan u + c$

Question 5 continued

$= \frac{1}{16}\left[\tan u - u\right] + c$

need in terms of 'x':

$= \frac{1}{16}\left[\tan(\tan^{-1}(4x)) - \tan^{-1}(4x)\right] + c$

$= \frac{1}{16}\left(4x - \tan^{-1}(4x)\right) + k$

expand $\frac{1}{16}$ in

$= \frac{1}{4}x - \frac{1}{16}\tan^{-1}(4x) + k$

$\therefore$ Subbing into I.B.P

Ans $= \frac{1}{2}x^2\tan^{-1}(4x) - \frac{1}{2}\left(\frac{1}{4}x - \frac{1}{16}\tan^{-1}(4x)\right) + k$

$\boxed{= \frac{1}{2}x^2\tan^{-1}(4x) - \frac{1}{8}x + \frac{1}{32}\tan^{-1}(4x) + k}$

(c) subbing into formula for mean value of a function

$\overline{f(x)} = \frac{1}{b-a}\int_a^b f(x)\,dx$

where $b = \sqrt{3}/4$, $a = 0$ and $f(x) = x\tan^{-1}(4x)$

$= \frac{1}{\frac{\sqrt{3}}{4} - 0}\int_0^{\sqrt{3}/4} x\tan^{-1}(4x)\,dx$

know indefinite integration from (b) - evaluate at LIMITS

$= \frac{1}{\sqrt{3}/4}\left[\frac{1}{2}x^2\tan^{-1}(4x) - \frac{1}{8}x + \frac{1}{32}\tan^{-1}(4x)\right]_0^{\sqrt{3}/4}$

$= \frac{4}{\sqrt{3}}\left\{\left[\frac{1}{2}\left(\frac{\sqrt{3}}{4}\right)^2\tan^{-1}(\sqrt{3}) - \frac{1}{8}\left(\frac{\sqrt{3}}{4}\right) + \frac{1}{32}\tan^{-1}(\sqrt{3})\right] - \left[0+0+0\right]\right\}$

$= \frac{4}{\sqrt{3}}\left\{\frac{\cancel{3}}{32}\left(\frac{\pi}{\cancel{3}}\right) - \frac{\sqrt{3}}{32} + \frac{\pi}{96}\right\}$

$= \frac{4}{\sqrt{3}}\left(\frac{\pi}{32} - \frac{\sqrt{3}}{32} + \frac{\pi}{96}\right)$

collect πs

$= \frac{4}{\sqrt{3}}\left(\frac{\pi}{24} - \frac{\sqrt{3}}{32}\right)$ $\quad = \frac{\sqrt{3}}{3}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right)$

rationalise coefficient

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA

Question 5 continued

expand

$$= \frac{\sqrt{3}\pi}{18} - \frac{3}{24}$$

or get common denominator

$$\boxed{= \frac{\sqrt{3}}{72}\left(4\pi - 3\sqrt{3}\right)}$$

Reminders:

Students find fractions tough as fractions can be so many types.

Check first (and throughout the question) if you can simplify by:

- using basic indices rules to simplify and expand brackets.
  - $x^a \times x^b = x^{a+b}$
  - $\frac{x^a}{x^b} = x^{a-b}$
  - $\frac{3}{5x}$ means $\frac{3}{5}x^{-1}$.
  - $(\sqrt[b]{x})^a$ or $\sqrt[b]{x^a} = x^{\frac{a}{b}}$
- Factorising and maybe cancel first
- Is there a single term in denominator?

  split fractions using $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$ or $(a+b)c^{-1}$

Then ask yourself:

1. **Is it an easy power type?** $\int x^n dx = \frac{x^{n+1}}{n+1}$
2. **Is it ln (natural logarithm)?** Form $\int \frac{f'(x)}{f(x)} dx$

   To recognize these, the power in the denominator is (almost always) 1. When you bring the denominator up to the numerator using negative power indices rule you get a power of -1. By adding one to the power and dividing it, you'll end up dividing by zero which you can't do

   $$\int \frac{f'(x)}{f(x)} dx = \ln f(x) + C$$

   Method: copy $\ln$(denominator). Remember ignore then differentiate to check you get what is inside the integral - correct with numbers only, not variables and only correct by multiplying or dividing. We can ignore the pink part since the derivative 'pops' out when we differentiate and we know when we differentiate our answer it must be what is inside the integral).
3. **Is it bring up and harder power type?** Bring the power up and becomes the form $\int f'(x) f(x)^n \, dx = \frac{f(x)^{n+1}}{n+1} + C$

   Recognisable by a power in the denominator other than

   $\int \frac{4x}{(2x^2-1)^3} = \int 4x(2x^2-10)^{-3} \, dx$ etc
4. **Is it Partial fractions!** Recognisable by products in the denominator.

   Form 1 $\frac{\dots}{(cx+d)(ex+f)} = \frac{A}{cx+d} + \frac{B}{ex+f}$

   Form 2 $\frac{\dots}{(dx+e)(fx+g)^2} = \frac{A}{dx+e} + \frac{B}{fx+g} + \frac{C}{(fx+g)^2}$

   (only advanced courses have this form)

   Form 3 $\frac{\dots}{(dx+e)(fx^2+g)} = \frac{A}{dx+e} + \frac{Bx+C}{fx^2+g}$
5. **Is it divide first?** Recognisable by **two or more terms** in the denominator and also where we have the matching highest powers in both numerator and denominator or a higher power in the numerator
6. **Rewriting/adapting fraction in a clever way (split up the numerator to get two fractions)**
7. **Is it inverse trig? (may need to complete the square first)** Either use the inverse trig results below or use a trig substitution

$$\int \frac{1}{\sqrt{a^2 - (bx)^2}} dx = \frac{1}{b}\sin^{-1}\left(\frac{bx}{a}\right) + C$$

$$\int -\frac{1}{\sqrt{a^2 - (bx)^2}} dx = \frac{1}{b}\cos^{-1}\left(\frac{bx}{a}\right) + C$$

$$\int \frac{1}{a^2 + (bx)^2} dx = \frac{1}{ab}\tan^{-1}\left(\frac{bx}{a}\right) + C$$

(Total for Question 5 is 10 marks)

6.

$$\mathbf{M} = \begin{pmatrix} k & 5 & 7 \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{pmatrix} \quad \text{where } k \text{ is a constant}$$

(a) Given that $k \neq 4$, find, in terms of k, the inverse of the matrix $\mathbf{M}$.

(4)

(b) Find, in terms of p, the coordinates of the point where the following planes intersect.

$$2x + 5y + 7z = 1$$
$$x + y + z = p$$
$$2x + y - z = 2$$

(3)

(c) (i) Find the value of q for which the following planes intersect in a straight line.

$$4x + 5y + 7z = 1$$
$$x + y + z = q$$
$$2x + y - z = 2$$

(ii) For this value of q, determine a vector equation for the line of intersection.

(7)

(a) remembering the steps to finding the inverse of a matrix

step 1: find det(M)

$$= k\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} - 5\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} + 7\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}$$

$$= k(-1-1) - 5(-1-2) + 7(1-2)$$

$$= k(-2) - 5(-3) + 7(-1)$$

expand brackets

$$= -2k + 15 - 7$$

$$= -2k + 8 \text{ or } 8 - 2k$$

step 2: find matrix of minors i.e each element replaced by the det. of the 2×2 matrix left after all elements in the rows and columns corresponding to that chosen element are deleted

$$M_{minors} = \begin{pmatrix} -2 & -3 & -1 \\ -5-7 & -k-14 & k-10 \\ 5-7 & k-7 & k-5 \end{pmatrix} = \begin{pmatrix} -2 & -3 & -1 \\ -12 & -k-14 & k-10 \\ -2 & k-7 & k-5 \end{pmatrix}$$

step 3: matrix of cofactors i.e change the sign of elements with '-ve'

$$\begin{pmatrix} \overset{+}{a} & \overset{-}{b} & \overset{+}{c} \\ \overset{-}{d} & \overset{+}{e} & \overset{-}{f} \\ \overset{+}{g} & \overset{-}{h} & \overset{+}{i} \end{pmatrix} \qquad C = \begin{pmatrix} -2 & 3 & -1 \\ 12 & -k-14 & 10-k \\ -2 & 7-k & k-5 \end{pmatrix}$$

P 6 2 6 7 1 A 0 2 0 2 8

step 4 : need C^T i.e keep main diagonal and swap positions of highlighted

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad C^T = \begin{pmatrix} -2 & 3 & -2 \\ 12 & -k-14 & 7-k \\ -1 & 10-k & k-5 \end{pmatrix}$$

step 5 : $M^{-1} = \frac{1}{\det(M)} C^T$

$$\Rightarrow M^{-1} = \frac{1}{8-2k} \begin{pmatrix} -2 & 3 & -2 \\ -12 & -k-14 & 7-k \\ -1 & 10-k & k-5 \end{pmatrix}$$

(b) WAY 1: using matrices

notice we're given a system of linear equations - hence need to find the p.o.i i.e solve for $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ → using general formula for matrix equations: $Mx = y$ - splitting into matrix of coefficients, variables and constants

$$\begin{pmatrix} 2 & 5 & 7 \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ p \\ 2 \end{pmatrix}$$

and solve for $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ ∴ multiply LHS of each side of the equation by M^{-1}

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & 5 & 7 \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ p \\ 2 \end{pmatrix}$$

and seeing how, comparing above inverse matrix to that in part (a), $k=2$ - subbing into Ans

$$M^{-1} = \frac{1}{4} \begin{pmatrix} -2 & 12 & -2 \\ 3 & -2-14 & 7-2 \\ -1 & 10-2 & 2-5 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{4} \begin{pmatrix} -2 & 12 & -2 \\ 3 & -16 & 5 \\ -1 & 8 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ p \\ 2 \end{pmatrix}$$

RHS matrix multiplication - "rows into columns"

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{4} \begin{pmatrix} -2 + 12p - 4 \\ 3 - 16p + 10 \\ -1 + 8p - 6 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} + 3p - 1 \\ \frac{3}{4} - 4 + \frac{5}{2} \\ -\frac{1}{4} + 2p - \frac{3}{2} \end{pmatrix}$$

Question 6 continued

$$\Rightarrow \boxed{\begin{aligned} x &= 3p - 3/2 \\ y &= \frac{13}{4} - 4p \\ z &= 2p - 7/4 \end{aligned}}$$

WAY 2: algebraically - solving 3 variable simultaneous equations for $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$2x + 5y + 7z = 1$ - ①
$x + y + z = p$ - ②
$2x + y - z = 2$ - ③

eg. elim. 'x' from ① and ②

① − 2×②
$$\begin{aligned} 2x + 5y + 7z &= 1 \\ -\ 2x + 2y + 2z &= 2p \\ \hline 3y + 5z &= 1 - 2p \end{aligned}$$ - ④

elim. 'x' from ② and ③

②×2 − ③
$$\begin{aligned} 2x + 2y + 2z &= 2p \\ -\ 2x + y - z &= 2 \\ \hline y + 3z &= 2p - 2 \end{aligned}$$ - ⑤

elim. 'y' from ④ and ⑤

3×⑤ − ④
$$\begin{aligned} 3y + 9z &= 6p - 6 \\ -\ 3y + 5z &= -2p + 1 \\ \hline 4z &= 8p - 7 \end{aligned}$$
$\div 4 \qquad \div 4$

$$\boxed{z = 2p - \frac{7}{4}}$$

sub this into any of ④ or ⑤ eg. into ⑤

$$y + 3\left(2p - \frac{7}{4}\right) = 2p - 2$$

expand

$$y + 6p - \frac{21}{4} = 2p - 2$$

$$\Rightarrow \boxed{y = -4p + \frac{13}{4}}$$

sub this into any of ①, ② or ③ eg. ②

$$x + \frac{13}{4} - 4p + 2p - \frac{7}{4} = p$$

$$\Rightarrow x = 3p - 6/4$$

$$\boxed{x = 3p - 3/2}$$

(c)(i) METHOD 1: checking consistency

recognising that for the planes to intersect in a straight line - $\det(A) = 0$

and the system of linear equations must have infinitely many solutions meaning they must be consistent with each other

∴ if we elim. one of the x, y, z variables to form three equations these would have to be consistent with one another

$$4x+5y+7z=1 \quad -①$$
$$x+y+z=q \quad -②$$
$$2x+y-z=2 \quad -③$$

WAY 1: elim. 'z'

$7\times② - ①$
$$7x+7y+7z=7q$$
$$-\ \ 4x+5y+7z=1$$
$$3x+2y=7q-1 \quad -④$$

$② + ③$
$$x+y+z=q$$
$$+\ \ 2x+y-z=2$$
$$3x+2y=q+2 \quad -⑤$$

we know the resulting 2 equations have to be consistent i.e their RHS has to be equal

$\Rightarrow 7q-1=q+2$

$\Rightarrow 6q=3$

$\div 6 \quad \div 6$

$\boxed{q=\frac{1}{2}}$

WAY 2: elim. 'x'

$① - 4\times②$
$$4x+5y+7z=1$$
$$-\ \ 4x+4y+4z=4q$$
$$y+3z=1-4q \quad -④$$

$2\times② - ③$
$$2x+2y+2z=2q$$
$$-\ \ 2x+y-z=2$$
$$y+3z=2q-2 \quad -⑤$$

we know that the resulting 2 equations have to be consistent i.e RHS has to be equal

$1-4q=2q-2$

$\Rightarrow 6q=3$

$\div 6 \quad \div 6$

$\Rightarrow \boxed{q=\frac{1}{2}}$

WAY 3: elim. 'y'

$5\times② - ①$
$$5x+5y+5z=5q$$
$$-\ \ 4x+5y+7z=1$$
$$x-2z=5q-1 \quad -④$$

$③ - ②$
$$2x+y-z=2$$
$$-\ \ x+y+z=q$$
$$x-2z=2-q \quad -⑤$$

we know that the resulting 2 equations have to be consistent i.e RHS has to be equal

$1-4q=2q-2$

$\Rightarrow 6q=3$

$\div 6 \quad \div 6$

$\Rightarrow \boxed{q=\frac{1}{2}}$

METHOD 2: finding p.o.i of the planes

first trying to solve ① and ② simultaneously (doesn't involve 'q') eg. $\boxed{\text{let } z=0}$

$$4x+5y=1 \quad -①$$
$$2x+y=2 \quad -③$$

elim. x

$① - 2\times③$
$$4x+5y=1$$
$$-\ \ 4x+2y=4$$
$$3y=-3$$
$$\div 3 \quad \div 3$$

$\boxed{y=-1}$ – subbing into any of ① or ③ to get 'x' eg. ①

$$4x+5(-1)=1$$

expand brackets

$$4x=6$$

$\Rightarrow \boxed{x=\frac{6}{4}=1.5}$

because 'intersect in a straight line' means all equations are consistent (at least one set of (x, y, z) that satisfies all equations simultaneously) – subbing this into ② to get 'q'

$$1.5 + (-1) + 0 = q$$

$$\Rightarrow \boxed{q = 0.5 = 1/2}$$

(ii) METHOD 1: finding two different coordinates that lie on the line of intersection and finding the vector equation through them

eg. ctd. from part (c)(i) WAY 1, have

$$3x + 2y = 7q - 1 \quad -④$$

$$3x + 2y = q + 2 \quad -⑤$$

but subbing $q = 1/2$ into any of ④ or ⑤, eg. ④

$$3x + 2y = 2.5 \quad -④$$

if we let $\boxed{x = 0}$,

$$\Rightarrow 2y = 2.5$$

$$\div 2 \qquad \div 2$$

$$\Rightarrow \boxed{y = 1.25}$$ – subbing into any ①, ②, ③ eg. ③

$$2(0) + 1.25 - z = 2$$

$$\Rightarrow \boxed{z = -0.75}$$

$\therefore$ one coordinate is $\begin{pmatrix} 0 \\ 1.25 \\ -0.75 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 5/4 \\ -3/4 \end{pmatrix}$

now let $\boxed{y = 0}$

$$\Rightarrow 3x = 2.5$$

$$\div 3 \qquad \div 3$$

$$\boxed{x = 5/6}$$

sub into any of ①, ②, ③ eg. ③

$$2(5/6) + 0 - (z) = 2$$

$$\Rightarrow \frac{5}{3} - z = 2$$

$$\Rightarrow \boxed{z = -1/3}$$

$\therefore$ second coordinate is $\begin{pmatrix} 5/6 \\ 0 \\ -1/3 \end{pmatrix}$

now given the 2 coordinates, can work out the vector parametric equation through them – in the form $a + \lambda b$

need 'b' $= \overrightarrow{AB} = \begin{pmatrix} 5/6 \\ 0 \\ -1/3 \end{pmatrix} - \begin{pmatrix} 0 \\ 5/4 \\ -3/4 \end{pmatrix}$

$$= \begin{pmatrix} 5/6 \\ -5/4 \\ 5/12 \end{pmatrix}$$

and any 'a' (A or B)

... A :

$$\boxed{r = \begin{pmatrix} 0 \\ 5/4 \\ -3/4 \end{pmatrix} + \lambda \begin{pmatrix} 5/6 \\ -5/4 \\ 5/12 \end{pmatrix}}$$

or ... B :

$$\boxed{r = \begin{pmatrix} 5/6 \\ 0 \\ -1/3 \end{pmatrix} + \lambda \begin{pmatrix} 5/6 \\ -5/4 \\ 5/12 \end{pmatrix}}$$

NOTE : can do the same (i.e make $x=0$, then $y=0$) for all the ④ and ⑤s of WAY 2 and WAY 3 to get the two coordinates

METHOD 2 : trying to fill in the general Cartesian equation for vector line :

$$\lambda = \frac{x-a}{b} = \frac{y-a}{b} = \frac{z-a}{b}$$

ctd. from part (c)(i) METHOD 1, WAY 1

realise how with $q = 1/2$, ④ becomes :

$$3x + 2y = 5/2$$

$\times 2$ $\times 2$

$$6x + 4y = 5$$

rearrange for 'y'

$$4y = 5 - 6x$$

$\div 4$ $\div 4$

$$y = \frac{5-6x}{4}$$

now we want a similar expression for 'z' in terms of 'x' : elim. 'y'

eg. ③ - ②

$$\begin{array}{r} 2x + y - z = 2 \\ - \quad x + y + z = q \\ \hline x - 2z = 2 - q \end{array}$$

subbing $q = 1/2$ in

$$x - 2z = 3/2$$

$\times 2$ $\times 2$

$$2x - 4z = 3$$

make 'z' the subject

$$4z = 2x - 3$$

$\div 4$ $\div 4$

$$\Rightarrow z = \frac{2x-3}{4}$$

as 'y' and 'z' are expressed in terms of 'x' - we can write the vector line parametrically

i.e let $x = \lambda$

$$y = \frac{5-6(\lambda)}{4}$$

$$z = \frac{2\lambda - 3}{4}$$

and rearranging both - now for λ

Question 6 continued

... y :

$4y = 5 - 6\lambda$

$6\lambda = 5 - 4y$

$\div 6 \quad \div 6$

$\lambda = \frac{5-4y}{6}$

... z :

$4z = 2\lambda - 3$

$\Rightarrow 2\lambda = 4z + 3$

$\div 2 \quad \div 2$

$\lambda = \frac{4z+3}{2}$

subbing into cartesian equation

$$x = \frac{5-4y}{6} = \frac{4z+3}{2} = \lambda$$

$$\Rightarrow x = -\frac{2}{3}y + \frac{5}{6} = 2z + \frac{3}{2} = \lambda$$

realising need single 'x', 'y', 'z' terms ($\div -2/3$, $\div 2$)

$$\frac{x-0}{1} = \frac{-\frac{2}{3}y + 5/6}{-2/3} = \frac{2z + 3/2}{2} = \lambda$$

$$\Rightarrow \frac{x-0}{1} = \frac{y - 5/4}{-3/2} = \frac{z + 3/4}{1/2} = \lambda$$

$\Rightarrow a$ = numerator negated $= \begin{pmatrix} 0 \\ 5/4 \\ -3/4 \end{pmatrix}$

b = denominator $= \begin{pmatrix} 1 \\ -3/2 \\ 1/2 \end{pmatrix}$

$$\boxed{\therefore r = \begin{pmatrix} 0 \\ 5/4 \\ -3/4 \end{pmatrix} + t \begin{pmatrix} 1 \\ -3/2 \\ 1/2 \end{pmatrix}}$$

(Total for Question 6 is 14 marks)

7.

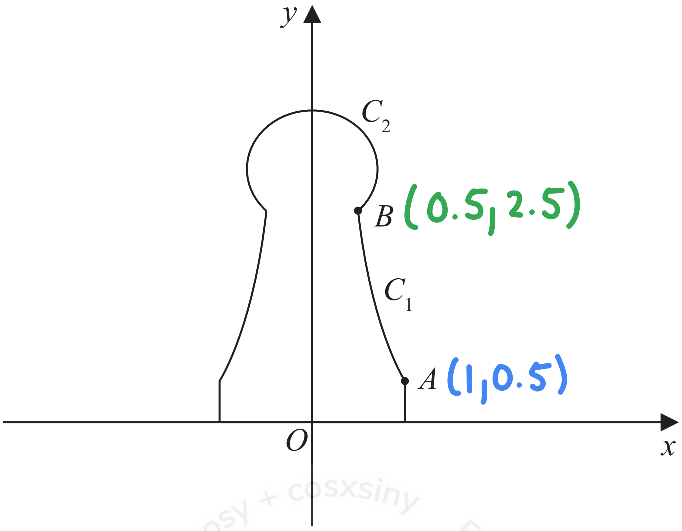


Figure 1

A student wants to make plastic chess pieces using a 3D printer. Figure 1 shows the central vertical cross-section of the student's design for one chess piece. The plastic chess piece is formed by rotating the region bounded by the y-axis, the x-axis, the line with equation $x = 1$, the curve C_1 and the curve C_2 through 360° about the y-axis.

The point A has coordinates (1, 0.5) and the point B has coordinates (0.5, 2.5) where the units are centimetres.

The curve C_1 is modelled by the equation

$$x = \frac{a}{y + b} \qquad 0.5 \leqslant y \leqslant 2.5$$

(a) Determine the value of a and the value of b according to the model.

(2)

The curve C_2 is modelled to be an arc of the circle with centre (0, 3).

(b) Use calculus to determine the volume of plastic required to make the chess piece according to the model.

(9)

(a) recognising that point A lies on C_1 and point B lies on C_1 ∴ subbing these in turn to get 'a' and 'b'

A : (1, 0.5) B (0.5, 2.5)

$1 = \frac{a}{0.5 + b}$ $0.5 = \frac{a}{2.5 + b}$

$\Rightarrow 0.5 + b = a$ $0.5(2.5 + b) = a$

$\Rightarrow a - b = 0.5$ -① expand brackets

$1.25 + 0.5b = a$

$\Rightarrow a - 0.5b = 1.25$ -②

P 6 2 6 7 1 A 0 2 4 2 8

Question 7 continued

solving these simultaneously – calc equation solver or elim. 'a'

②-① $0.5b = 0.75$

$\div 0.5 \quad \div 0.5$

$\boxed{b = 1.5}$ – sub this into any of ① or ② for 'a'

$a = 0.5 + 1.5$

$\Rightarrow \boxed{a = 2}$

(b) key bit is realising that this is a modelling with volumes of revolution question where to determine the volume of the plastic required to make the whole chesspiece would require us to add 3 separate volumes together

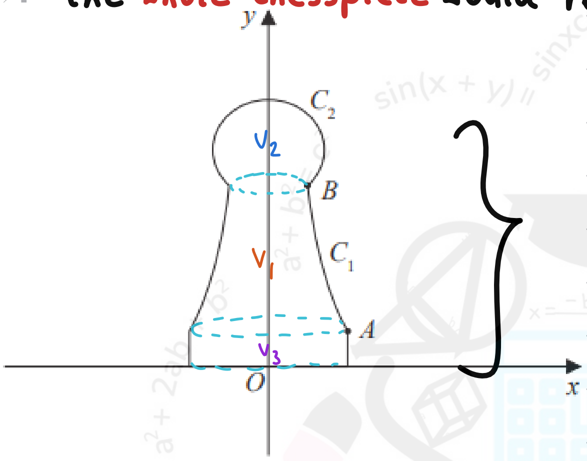


Figure 1

$V_{\text{chess piece}} = V_1 + V_2 + V_3$

... first: V_1 :

remembering formula for volume of revolution about the y-axis:

$$\pi \int_\alpha^\beta x^2 \, dy$$

$$V_1 = \pi \int_{0.5}^{2.5} \left(\frac{\text{“}2\text{”}}{y + \text{“}1.5\text{”}} \right)^2 dy$$

factorise the 4 out

$$= 4\pi \int_{0.5}^{2.5} \frac{1}{(y+1.5)^2} \, dy \quad \text{using index laws} \quad = 4\pi \int_{0.5}^{2.5} (y+1.5)^{-2} \, dy$$

integrate

$$4\pi \left[-(y+1.5)^{-1} \right]_{0.5}^{2.5}$$

$$= 4\pi \left\{ \left[-(2.5+1.5)^{-1} \right] - \left[-(0.5+1.5)^{-1} \right] \right\}$$

$$= 4\pi \left(-\frac{1}{4} + \frac{1}{2} \right) = 4\pi \left(\frac{1}{4} \right) = \pi \text{ cm}^3$$

...next: V_2-requires us to find the volume of revolution of an arc circle - finding its cartesian equation - generally $(x-a)^2+(y-b)^2=r^2$

centre: $(0,3)$

radius = eg. $\sqrt{(0.5-0)^2+(2.5-3)^2}$

$= \sqrt{(0.5)^2+(-0.5)^2}$

$= \sqrt{2}/2$

$\therefore x^2+(y-3)^2=1/2$

and now working out the volume of revolution of the circle's cartesian equation - sub it into $\pi\int_\alpha^\beta x^2\,dy$

$$= \pi\int_{2.5}^{\alpha}\left(0.5-(y-3)^2\right)dy \quad ; \quad \alpha = 3+\frac{\sqrt{2}}{2}$$

integrate

$$= \pi\left[0.5y-\frac{1}{3}(y-3)^3\right]_{2.5}^{\alpha}$$

$$= \pi\left\{\left[0.5\alpha-\frac{1}{3}(\alpha-3)^3\right]-\left[\pi(0.5(2.5)-\frac{1}{3}(2.5-3)^3\right]\right.$$

$$= \pi\left(\frac{6+\sqrt{2}}{4}-\frac{\sqrt{2}}{12}\right)-\pi\left(\frac{31}{24}\right)$$

$$= \pi\left(\frac{9+\sqrt{2}}{6}\right)-\frac{31}{24}\pi$$

$$= \frac{5+4\sqrt{2}}{24}\pi \text{ cm}^3$$

...finally: V^3 - see straight lines; using volume of cylinder $=\pi r^2h$

$$\pi(1)^2(0.5)$$

$$= \frac{\pi}{2}\text{cm}^3$$

$\Rightarrow$ TOTAL VOLUME $= \pi + \frac{5+4\sqrt{2}}{24}\pi + \frac{\pi}{2}$

$$= \pi\left(1+\frac{5+4\sqrt{2}}{24}+\frac{1}{2}\right) = \pi\left(\frac{41+4\sqrt{2}}{24}\right)\text{cm}^3$$

Question 7 continued

$\approx \boxed{6.11 cm^3 (3 s.f)}$

Question 7 continued

(Total for Question 7 is 11 marks)

TOTAL FOR PAPER IS 75 MARKS